# Mathematical Escapades

## Volume 1

SRI Mathematics Challenge 2015
Problems and Solutions

Includes hints and additional exercises.

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## Preface

In June 2015, the Simplicity Research Institute held its first online mathematics challenge for high-school students resident in Singapore. This volume contains problems from that event, hints for their solution, and the solutions. We have also included additional exercises for enthusiasts to test themselves. We hope you find the material here useful.

Singapore, April 2016.

# Chapter 1

# SRIMC 2015 Problems

1. Let

$$x = \left( \frac{pa^n + qb^n + rc^n}{pd^n + qe^n + rf^n} \right)^{\frac{1}{n}} , \tag{1.1}$$

with $n$ a positive integer, and the other letters in the equation representing real positive (non-zero) variables.

If $x$ is identically equal to $\frac{a}{d}$, whatever values the other variables take, express the ratios $\frac{b}{e}$ and $\frac{c}{f}$ in their simplest forms in terms of $x$ and $n$.

2. Let

$$A = (\log_e X)(\log_{10} Y) , \tag{1.2}$$

and

$$B = (\log_e Y)(\log_{10} X) . \tag{1.3}$$

Is there any relation between $A$ and $B$ for arbitrary positive $X$ and $Y$? Justify your answer.

3. Consider the infinite series

$$\sqrt{1 + \sqrt{2 + \sqrt{1 + \sqrt{2 + \sqrt{1 + ...}}}}}$$

where "..." implies (an infinite) continuation of the pattern seen in the first few terms, and where $\sqrt{.}$ is the usual square-root sign. The series converges to a finite value $S$.

(a) Show that $(S^2 - 1)^2 = 2 + S$.

(b) Hence, or otherwise, prove that $S$ cannot be rational. (You do not need to determine the value of $S$).

4. If $1 - x + x^2 = 0$, determine the exact value of $y = (x^2 + 1)(x - 1)$ without using software. Express your answer in its simplest form. (You do not need to determine the value of $x$.)

5. Let $A$ be the area of a square inscribed in a semi-circle of radius $R$. Let $B$ be the area of a square inscribed in a circle of radius $R$. Find the ratio $B/A$ in simplest terms.

6. A triangle is inscribed in a unit square (that is, a square with each side of length one). What is the maximum possible area of the triangle? Justify your answer.

7. Let $a$ and $b$ be the two perpendicular sides of a right-angled triangle, and let $h$ be the height from the hypotenuse to the right-angled vertex. Show that the sum

$$\frac{1}{a^2} + \frac{1}{b^2}$$

may be written solely in terms of $h$. Express your answer in simplest terms.
(Note: The height $h$ is perpendicular to the hypotenuse.)

8. $a, b$ and $c$ are the sides of a right-angled-triangle, with $c$ the hypotenuse.
Three regular heptagon's are constructed: The first with sides of length $a$, the second with sides of length $b$, and the third with sides of length $c$. Denote the areas of the heptagon's by $A_a$, $A_b$ and $A_c$ respectively. Show that

$$A_a + A_b = A_c.$$

(Note: A regular heptagon has seven sides of equal length).

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# Chapter 2

# Hints for Problems

1. Substitute $a = xd$ into the given equation and rewrite it in the form

$$qS + rT = 0\,,$$

where $S$ depends on $x, b, e, n$ and $T$ depends on $x, c, f, n$. Next, recall the crucial phrase in the question: *"If $x$ is identically equal to $\frac{a}{d}$, whatever values the other variables take"*, to draw your conclusion on the values of $S$ and $T$, and solve the problem.

2. The logarithms in $A$ and $B$ have their bases interchanged. This suggests using the change of base formula

$$\log_b c = \frac{\log_a c}{\log_a b}\,. \tag{2.1}$$

3. Set the expression equal to $S$.

   (a) The result suggests the obvious next step: Square both sides to find $S^2$, and manipulate the resulting equation further to get $S$ again on the right-hand-side.

   (b) Prove by contradiction. Assume $S = a/b$ with $a$ an integer (possibly negative) and $b$ a positive integer, and with $a, b$ being relatively prime (that is, with no common factors). Substitute this into the expression of the previous part and remove all denominators by cross-multiplying. Next look at the divisibility of the expression by $a$ and $b$ to deduce possible constraints on $a$ and $b$. Finally test to see if those values of $a$ and $b$ solve for $S$.

4. Directly solving for $x$ gives a complex number which is tedious to manipulate. There are several other ways to do this. For example,

you can replace $1+x^2$ in $y$ by $x$ (using the original equation for $x$), simplify and proceed.

5. Draw the figures and use Pythagoras' theorem twice.

6. You can do this in two parts. First assume two vertices of the triangle are on one side of the square. Determine the maximum possible area of the triangle. Second, assume that no two vertices are on the same side, and show that such a triangle will have a lower area than the maximum from the first part.

7. Write the given sum over a common denominator. You should now recognise some of the expressions, and with the help of a figure, deduce the answer.

8. First show that if a triangle $P$ is similar to triangle $Q$, but with each length of $P$ being $x$ times the corresponding length in $Q$, then the area of $P$ is $x^2$ times the area of $Q$. Next, deduce the same for polygons by dividing the polygons into triangles.

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# Chapter 3

# Solutions to Problems

Following the hints in the previous chapter, we proceed as follows.

1.

$$x^n(pd^n + qe^n + rf^n) \equiv pd^n x^n + qb^n + rc^n \tag{3.1}$$

$$\Rightarrow q(x^n e^n - b^n) + r(x^n f^n - c^n) \equiv 0\,. \tag{3.2}$$

Since the last equation is identically equal to zero for all $q$ and $r$, setting $q = 0$ first implies $c/f = x$, and then when $q \neq 0$ we get $b/e = x$.

Answer: $b/e = c/f = x$.

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2. Dividing Eq.(1.2) by Eq.(1.3), and using Eq.(2.1) we get

$$\begin{aligned} \frac{A}{B} &= \frac{(\log_e X)(\log_{10} Y)}{(\log_e Y)(\log_{10} X)} \\ &= (\log_Y X)(\log_X Y) \\ &= 1\,. \end{aligned}$$

Answer: $A = B$.

---

3.

$$\begin{aligned} S &= \sqrt{1+\sqrt{2+\sqrt{1+\sqrt{2+\sqrt{1+\ldots}}}}} \\ S^2 &= 1+\sqrt{2+\sqrt{1+\sqrt{2+\sqrt{1+\ldots}}}} \\ (S^2-1)^2 &= 2+S \\ S^4-2S^2-S-1 &= 0. \end{aligned} \tag{3.3}$$

The third line above is the answer for the first part. Assume now that $S$ is rational and set $S = a/b$ in the last equation, with $a, b$ relatively prime integers and $b > 0$. Then

$$a(a^3 - 2b^2a - b^3) = b^4\,. \tag{3.4}$$

This implies that $b^4$ is divisible by $a$, but since $a$ and $b$ are relatively prime, this means (using the unique prime factorisation of integers) that $b$ is divisible by $a$. So $a = \pm 1$. Similarly, by re-arranging Eq.(3.4) we deduce that $b = 1$. So if $S$ is rational, the only candidates are $S = \pm 1$. But direct substitution into Eq.(3.4) shows neither to be a solution. This contradicts our initial assumption. Hence $S$ must be irrational.

---

4. We have

$$1 - x + x^2 = 0 \tag{3.5}$$

$$\text{or} \quad x^2 + 1 = x\,. \tag{3.6}$$

Therefore, using the above two expressions satisfied by $x$, we can simplify

$$\begin{aligned} y &= (x^2+1)(x-1) \\ &= x(x-1) \\ &= x^2 - x \\ &= -1\,. \end{aligned}$$

 

Alternatively, you can expand out $y$ to get $y = x^3 - x^2 + x - 1$ and use Eq.(3.5) multiplied by $x$ to simplify $y$ to $-1$.

Yet another way is to recognise the left-side of Eq.(3.5) as the sum of a geometric series and so get $x^3 = -1$, and therefore $y = -1$. (This method reveals $x$ to be the non-real cubic root of $-1$).

Answer: $y = -1$.

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5. This is a straightforward exercise which gives $A = 4R^2/5$ and $B = 2R^2$.

   Answer: $B/A = 5/2$.

---

6. Consider first the case where two vertices of the triangle are on one side. Then since the area of the triangle is base $\times$ height/2, its clear that in this case the maximum area occurs when those two vertices coincide with the adjacent vertices of the square (of side 1) and the third vertex is on the opposite edge, giving an area of 1/2.

   Next, consider the possibility where each vertex of the triangle lies on a different side. A typical situation is shown as triangle $ABC$ in the figure below where $D$ is one vertex of the square and line $EB$ is parallel to $DC$. Now slide point $C$ until it reaches $D$ to get the new triangle $ABD$. The area $BCG$ lost from the original triangle equals the area $DFG$ inside the new triangle. So the new triangle $ABD$ has a larger area, by $AFD$, than the original triangle $ABC$. With vertices $A$ and $D$ on one side, the rest of the argument is as in the first case. (The figure shows $A$ higher than $B$; for the reverse case, slide the point $C$ to the same side as $B$. Finally, if $A$ and $B$ are at the same height then clearly $AB$ can be pushed to one edge, increasing the area).

   Answer: Maximum area of triangle is 1/2 the area of the square.

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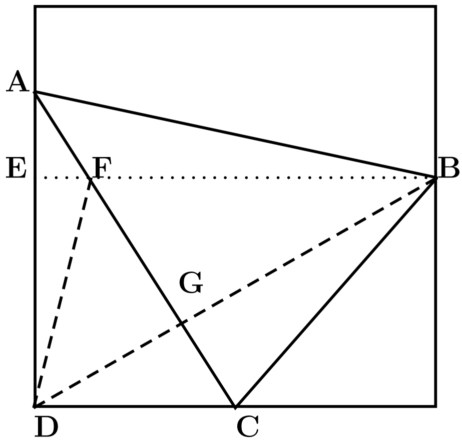


Figure 3.1: Triangle in a square of side 1.

7. Simplify the sum,

$$\frac{1}{a^2} + \frac{1}{b^2} = \frac{a^2 + b^2}{(ab)^2} \tag{3.7}$$

$$= \frac{c^2}{(hc)^2} \tag{3.8}$$

$$= \frac{1}{h^2}\,. \tag{3.9}$$

In Eq.(3.8) we used Pythagoras' Theorem and the formula for the area of the triangle: $ab/2 = hc/2$.

Answer: $1/h^2$.

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8. In the figure below, triangle $ABF$ is similar to $ACD$. So if $AF/AD = x$, then $BF/CD = x$. The dotted lies are the heights of the two triangles above their base. Triangles $ABG$ is similar to $ACE$, and again $BG/CE = AF/AD = x$. So the area of triangle $ABF$ is $x^2$ times the area of triangle $ACD$.

   Now, divide a regular heptagon into triangles, for example by drawing a line from the centre to each vertex. Then, using the above result for

 

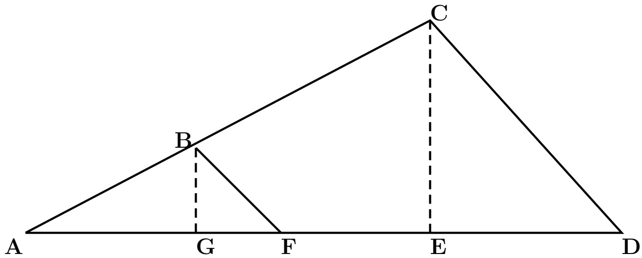


Figure 3.2: Similar triangles

triangles we deduce that the area of the heptagon scales as $x^2$ when each edge is scaled by a factor $x$.

Therefore $A_a = \left(\frac{a}{c}\right)^2 A_c$ and $A_b = \left(\frac{b}{c}\right)^2 A_c$, giving

$$\begin{aligned} A_a + A_b &= \frac{a^2 + b^2}{c^2} A_c \\ &= A_c \end{aligned}$$

upon using Pythagoras' theorem, $a^2 + b^2 = c^2$.

Answer: $A_a + A_b = A_c$.

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# Chapter 4

# Exercises

1. Fill in the details for the solutions in the last chapter.

2. If $\frac{a}{d} = \frac{b}{e} = \frac{c}{f} = x$, express $y$, given below, in its simplest form.

$$y = \left( \frac{pa^n + qb^n + rc^n}{pd^n + qe^n + rf^n} \right)^{\frac{1}{n}} , \tag{4.1}$$

   with $n$ a positive integer, and the other letters in the equation representing real positive (non-zero) variables.

3. Prove that $\sqrt{7} - \sqrt{3}$ is irrational.
   (Can you do this in more than one way?)

4. Without using a calculator, find the value of

$$\frac{\log_5 49}{\log_6 49} - \frac{\log_5 81}{\log_6 81} .$$

5. If $A = p^{\log_q r}$ and $B = r^{\log_q p}$, what is the relation, if any, between $A$ and $B$ for general $p, q$ and $r$.

6. Challenge: A string of length $l$ is used to construct the boundary of a triangle. Determine the maximum possible area of the triangle.

# Chapter 5

# Answers to Selected Exercises

2. $y = x$.

4. 0.

5. $A = B$

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**Did You Know?**

The Collatz Conjecture:

"Start with any positive integer $n$. If it is even, divide it by 2; if odd, multiply by 3 and add 1. Continue the procedure on each result. You will eventually reach 1".

For example: $17 \to 52 \to 26 \to 13 \to 40 \to 20 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1$.

But is the claim true for all positive integers? At time of writing, the conjecture remains unproven.

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