Handbook of Mathematics

For Students and Explorers

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Editor

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Handbook of Mathematics:
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MENU

This book is dedicated to one of my first teachers, my late aunt, Parpati.

Preface

In addition to providing reference for the usual topics covered in high-schools and colleges, this book includes eclectic titbits to stimulate enquiry and investigation.

Acknowledgements

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References

For details on the material in this handbook, please refer to the relevant entries in Wikipedia (www.wikipedia.org) or MathWorld (mathworld.wolfram.com).

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- Any non-zero number* may be written in **standard (scientific) notation** as $\pm A \times 10^p$ where $1 \leq A < 10$ and p is an integer.
- Some common prefixes:
  - nano: 10^{-9}.
  - micro: 10^{-6}.
  - milli: 10^{-3}.
  - kilo: 10^{3}.
  - mega: 10^{6}.
  - giga: 10^{9}.
- 1 centimetre (cm) = 10 millimetres (mm) .
- 1 metre (m) = 100 cm.
- 1 light-year $\approx 9.46 \times 10^{12}$ km.
- 1 hectare = $10,000$ m^2.
- 1 litre (L) = 1000 cm^3.
- 1 kilogram (kg) = 1000 grams (g).
- 1 tonne (t) = 1000 kg.
- 1 dozen = 12 units.
- 1 googol = 10^{100} units.

**Note: Unless otherwise stated, we will deal only with real numbers in this handbook.*

- A **rational number** can be written as the ratio of two integers, For example, $0.23 = 23/100$ is rational.
- Most numbers are **irrational**, that is, not rational. Examples are $\sqrt{2}$, π, and e (Euler's constant).
- Some numbers are suspected to be irrational but a proof is lacking at the time of writing. Examples are $\pi \pm e$, and π^e.
- $\pi \approx 3.142$.
- $\pi^2 \approx 9.87$.
- $e \approx 2.718$.
- $\sqrt{2} \approx 1.414$.
- $\sqrt{3} \approx 1.732$.
- $\sqrt{5} \approx 2.236$.
- ϕ (Golden ratio) ≈ 1.618.

Did You Know?

The first 25 digits of π:

3.141 592 653 589 793 238 462 643 3....

Greek letters are often used in mathematics. The lower case, and some upper case letters are indicated below.

- α alpha.
- β beta.
- γ gamma.
- Γ Gamma.
- δ delta.
- Δ Delta .
- ϵ epsilon.
- ζ zeta.
- η eta.
- θ theta.
- ι iota
- κ kappa
- λ lambda.
- Λ Lambda.
- μ mu.
- ν nu.
- ξ xi.
- o omicron.
- π pi.
- Π Pi.
- ρ rho.
- σ sigma.
- Σ Sigma .
- τ tau.
- υ upsilon.
- ϕ phi.
- χ chi.
- ψ psi.
- ω omega.
- Ω Omega.

Did You Know?

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e,$$

where e is Euler's constant.

- Prime numbers are positive integers that have exactly two factors: 1 and themselves.
- The first few primes are $2, 3, 5, 7, 11, 13, 17, 19, \ldots$.
- There are an infinite number of primes, as first shown by Euclid.
- An elementary (but slow) way to check for primality of a number N is to check if the number is divisible by primes less than $\sqrt{N}$.
- **Prime Factorisation Theorem**:
  Every integer has a unique decomposition into a product of its prime factors.
- There is no known explicit formula for the n-th prime.
- **Prime Number Theorem**:
  Let $\pi(n)$ count the number of primes up to n. For example, $\pi(7) = 4$. Then, as $n \to \infty$,

  $$\pi(n) \sim \frac{n}{\ln n},$$

  meaning that the ratio $\pi(n)/(n/\ln n)$ approaches 1 as $n \to \infty$. (Note: ln is the natural logarithm).

- A number is divisible by 2 if its last digit is even. For example, 26498 is even.
- A number is divisible by 3 if the sum of its digits is divisible by 3. For example, 123 is divisible by 3, but 431 is not.
- A number is divisible by 4 if the number represented by its last two digits is divisible by 4. For example, 26418 is not divisible by 4 because 18 is not.
- A number is divisible by 5 if its last digit is 0 or 5.
- A number is divisible by 9 if the sum of its digits is divisible by 9. For example, 126 is divisible by 9.
- Let L be the **least common multiple** of the natural numbers m and n, and let G be their **greatest common divisor**. Then $GL = mn$.
- $a^2 - b^2 = (a - b)(a + b)$.
- $a^3 \mp b^3 = (a \mp b)(a^2 \pm ab + b^2)$.
- $x^n - 1 = (x-1)(1+x+\cdots+x^{(n-2)} + x^{(n-1)})$.
- For n odd, $x^n + 1 = (x+1)(1-x+x^2-x^3+\cdots+x^{(n-1)})$.

- **Arithmetic Progression** (AP): The n-th term, a_n, of a sequence in arithmetic progression is given by $a_n = a_1 + (n-1)d$, where d is the common difference between consecutive terms.
- The sum, S_n, of n consecutive terms of an AP is given by $\frac{n}{2}(a_1 + a_n)$.
- **Geometric Progression** (GP): The n-th term, a_n, of a sequence in geometric progression is given by $a_n = a_1 r^{n-1}$ where $r = a_2/a_1$ is the common ratio between consecutive terms.
- The sum, S_n, of n consecutive terms of a GP is given by $\frac{a_1(r^n - 1)}{r-1}$.
- For a GP with $|r| < 1$, the infinite sequence has a convergent sum $S_\infty = \frac{a_1}{1-r}$.
- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$
- For n real and $|x| < 1$ we have the convergent series

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

- For a positive integer n,

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \dots$$
$$+ \binom{n}{r} a^{n-r} b^r + \dots + \binom{n}{n-1} a b^{n-1} + b^n$$
$$= \sum_{r=0}^{r=n} {}^nC_r \; a^{n-r} \; b^r \, .$$

- The **binomial coefficient** is defined by

$${}^nC_r \equiv \binom{n}{r} = \frac{n!}{r! \; (n-r)!}$$

with $n! \equiv n \times (n-1) \times (n-2) \times .... \times 2 \times 1$ and $0! \equiv 1$.

- The $(r+1)$-th term in the expansion is given by

$$\binom{n}{r} a^{n-r} b^r .$$

- Special Cases:

$$(a \pm b)^2 = a^2 \pm 2ab + b^2 \; .$$
$$(a \pm b)^3 = a^3 \pm 3a^2 b + 3ab^2 \pm b^3 \; .$$
$$(a \pm b)^4 = a^4 \pm 4a^3 b + 6a^2 b^2 \pm 4ab^3 + b^4 \; .$$

- The number of ordered arrangements of n distinguishable objects on a line is $n!$.
- The number of ordered arrangements of n objects on a line is

$$\frac{n!}{p!\ q!\ r!}\ ,$$

where there are p identical objects of one type, q identical objects of a second type, etc.

- The number of ordered arrangements of r objects (on a line), chosen from a collection of n distinguishable objects is

$$^{n}P_{r} \equiv \frac{n!}{(n-r)!}\ .$$

- The number of ways of choosing r objects from a collection of n distinguishable objects is

$$^{n}C_{r} \equiv \frac{n!}{r!\ (n-r)!}\ .$$

- **Pascal's Identity**: For $0 \leq r \leq n$,

$$\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}\ .$$

- If a variable y is **proportional to** another variable x, $y \propto x$, then $y = kx$ for some constant k. That is, the ratio y/x is a constant equal to k.

- If a variable y is **inversely proportional to** another variable x, $y \propto 1/x$, then $y = k/x$ for some constant k. That is, the product $y \cdot x$ is a constant equal to k.

- The fraction a/b corresponds to $100a/b$ **percent**. For example, the fraction $1/2$ is 50% while $3/2$ is 150%.

- Money deposited in a bank savings account typically earns interest. **Simple interest** at $R\%$ per annum on a principal of P dollars will yield $P(1 + r)$ dollars at the end of the year, where $r = R/100$.

- **Compound Interest**: The nominal annual interest of $R\%$ may be divided into N parts, and an interest of $(R/N)\%$ paid on the accumulated amount (principal plus interest) at the end of each $\frac{12}{N}$ months. So at the end of t years an initial savings of P would have grown to $P\left(1 + \frac{r}{N}\right)^{Nt}$ where $r = R/100$.

- Area of a triangle:
  $\frac{1}{2}b \times h$, where b is the base and h the height.
- Area of a parallelogram:
  $b \times h$, where b is the base and h the height.
- Area of a circle of radius r: πr^2.
- Circumference of a circle of radius r: $2\pi r$.
- Volume of pyramid or cone:
  $\frac{1}{3}$(area of base) $\times$ (vertical height).
- Volume of solid figure of constant cross-section:
  (area of cross-section) $\times$ (vertical height).
- Surface area of solid figure of constant cross-section:
  $2 \times$ (area of cross-section) $+$
  (perimeter of cross-section) $\times$ (vertical height).
- Volume of a sphere of radius r: $\frac{4}{3}\pi r^3$.
- Surface area of a sphere of radius r: $4\pi r^2$.
- Lateral surface area of cone: $\pi r l$, where l is the lateral height, and r the radius of the circle at the base. (The base area is πr^2).

Let a and b be two real numbers. Then the following means may be defined (restricting to positive numbers for the geometric mean).

- **Quadratic Mean Q**: $Q = \sqrt{\dfrac{a^2 + b^2}{2}}$ (also known as **root mean square**).
- **Arithmetic Mean A**: $A = \dfrac{a+b}{2}$.
- **Geometric Mean G**: $G = \sqrt{ab}$.
- **Harmonic Mean H**: It is determined by
$$\frac{2}{H} = \frac{1}{a} + \frac{1}{b}.$$
- The following relations hold between the means:
$$G = \sqrt{AH}.$$
$$Q \geq A \geq G \geq H.$$

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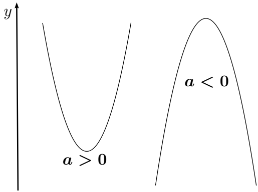
Let (a_1, a_2) and (b_1, b_2) be pairs of real numbers in the following.

- **Cauchy-Schwartz**: $(a_1b_1 + a_2b_2)^2 \leq (a_1^2 + a_2^2)(b_1^2 + b_2^2)$.
- **Triangle Inequality**: $\sqrt{(a_1+b_1)^2 + (a_2+b_2)^2} \leq \sqrt{a_1^2 + a_2^2} + \sqrt{b_1^2 + b_2^2}$.
- **Rearrangement Inequality**: If $a_2 \geq a_1$ and $b_2 \geq b_1$ then $a_2b_2 + a_1b_1 \geq a_2b_1 + a_1b_2$.
- **Isoperimetric Inequality**: A plane (closed) figure of area A and perimeter P satisfies $4\pi A \leq P^2$. This implies that the circle has the largest area for a given perimeter.
- $e^a \geq 1 + a$.
- **Bernoulli's Inequality**: $(1+a)^b \geq 1 + ab$ for $a \geq -1$ and $b \geq 1$.
- For $x > 0$, $x^x \geq \left(\frac{1}{e}\right)^{1/e}$.
- **Stirling's approximation**:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \text{ as } n \to \infty.$$

- The solution of the quadratic equation $ax^2 + bx + c = 0$, with $a \neq 0$ is given by the roots
$$x = \frac{-b \pm \sqrt{\Delta}}{2a} .$$
- $\Delta \equiv b^2 - 4ac$ is called the **discriminant**.
- The two **roots** are real if and only if $\Delta \geq 0$; the case $\Delta = 0$ corresponds to a repeated root.
- The orientation of the **parabola** $y(x) = ax^2 + bx + c$ is determined by the sign of a: When $a > 0$, the curve has a minimum point while for $a < 0$ it has a maximum.
- The **symmetry axis** of the parabola is at $x = \dfrac{-b}{2a}$.



- **Factorisation Theorem**:
For a polynomial $P(x)$, $P(\alpha) = 0$ if and only if $P(x) \equiv (x - \alpha)Q(x)$ for some polynomial Q.

- **Remainder Theorem**: If the polynomial $P(x)$ is divided by $(x - \alpha)$, the remainder is $P(\alpha)$. That is, $P(x) \equiv (x-\alpha)Q(x)+R$ with $R = P(\alpha)$.

- For a **cubic curve** $y = ax^3 + bx^2 + cx + d$, the sign of the leading coefficient determines its main shape. If $a > 0$, the curve rises upwards for large positive x and decreases for large negative x. In between, it might have a local minimum and a local maximum.

- **Partial fraction** decomposition of a rational function $P(x)/Q(x)$: First, use long division to reduce the degree of the numerator to below that of the denominator. Next, each factor of $(x - a)$ in $Q(x)$ would require a partial fraction $A/(x - a)$. If the factor is repeated in Q, for example $(x-a)^2$, then one uses two partial fractions $A_1/(x-a)$ and $A_2/(x-a)^2$ for that factor. If Q contains a term that cannot be factorised (using real numbers), for example $x^2 + x + 1$, then the partial fraction for that term is of the form $(Ax+B)/(x^2+x+1)$, that is, the numerator is one degree lower than the denominator.

For $a, b > 0$,

- $a^{-1} = \frac{1}{a}$.
- $a^0 = 1$.
- $a^{xy} = (a^x)^y$.
- $a^x a^y = a^{x+y}$.
- $(ab)^x = a^x b^x$.
- $a^{\frac{1}{2}} = \sqrt{a}$.
- $\sqrt{ab} = \sqrt{a}\,\sqrt{b}$.
- $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$.
- For $a > 1$, $y = a^x$ is positive and an increasing function of x along the real line; for $0 < a < 1$, a^x is a decreasing function of x.

Did You Know?

Euler's Identity:

$$e^{i\theta} = \cos\theta + i\sin\theta \ ,$$

where $i = \sqrt{-1}$.

The basic relation between exponents and logarithms is

$$y = b^x \Leftrightarrow x = \log_b y \ .$$

For $a, b > 0$, $a \neq 1$, $b \neq 1$ and $P, Q > 0$,

- $\log_b 1 = 0$.
- $\log_b PQ = \log_b P + \log_b Q$.
- $\log_b \dfrac{P}{Q} = \log_b P - \log_b Q$.
- $\log_b P^c = c \log_b P$.
- $\log_b P = \dfrac{\log_a P}{\log_a b}$.
- $\log_b a = \dfrac{1}{\log_a b}$.
- For $a > 1$ and $x > 0$, $y = \log_a x$ is an increasing function of x; it is positive for $x > 1$, negative for $x < 1$ and vanishes at $x = 1$.
- The graph of $y = \log_a x$ may be obtained by reflecting the exponential curve $y = a^x$ about the line $y = x$.
- The **natural logarithm**, $\log_e x$, where e is Euler's constant, is often denoted by $\ln x$.

- A $m \times n$ matrix A, multiplying from the left to a $n \times p$ matrix B, yields a $m \times p$ matrix C; that is $C = AB$. The c_{ij} element of C is obtained by multiplying the i-th row of A to the j-th column of B, term by term, and adding the pieces.

- The pair of **simultaneous equations** in the variables (x, y),

$$ax + by = e$$
$$cx + dy = f\,,$$

may be written as the matrix equation $MX = R$ with

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$
$$X = \begin{pmatrix} x \\ y \end{pmatrix}$$

and

$$R = \begin{pmatrix} e \\ f \end{pmatrix}.$$

- The **determinant** of the 2×2 matrix M is defined by det(M)$= ad - bc$.

- If det(M)$\neq 0$, one may define an **inverse matrix**

$$M^{-1} = \frac{1}{\det(\mathrm{M})} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

(continued →)

- If det(M)$= 0$ the matrix is termed **singular** and there is no inverse matrix.

- The inverse matrix satisfies

$$MM^{-1} = M^{-1}M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

  the last matrix being the **identity matrix**, usually denoted by the letter I.

- The solution of $MX = R$ is

$$X = M^{-1}R = \frac{1}{\det(\mathrm{M})} \begin{pmatrix} de - bf \\ -ce + af \end{pmatrix}.$$

Did You Know?

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- The **magnitude** of a vector $\vec{a} = a_x\vec{i} + a_y\vec{j} + a_x\vec{k}$ is $|\vec{a}| \equiv \sqrt{a_x^2 + a_y^2 + a_z^2}$ where i, j, k are orthonormal vectors, that is, $i \cdot i = j \cdot j = k \cdot k = 1$ and $i \cdot j = i \cdot k = j \cdot k = 0$.

- The **scalar** or **dot product** of two vectors is

  $$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} = a_x b_x + a_y b_y + a_z b_z = |\vec{a}||\vec{b}| \cos\theta \ ,$$

  where θ is the angle between $\vec{a}$ and $\vec{b}$.

- The **vector** or **cross product** of two vectors is

  $$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a} = |\vec{a}||\vec{b}| \sin\theta \ \vec{n} \ ,$$

  where $\vec{n}$ is the unit vector perpendicular to the plane defined by $\vec{a}$ and $\vec{b}$; its direction given by the **right-hand** rule (point your right-hand fingers in the direction of $\vec{a}$ and close them in the direction of $\vec{b}$, the thumb points along $\vec{n}$). In Cartesian coordinates, $\vec{a} \times \vec{b}$ is

  $$(a_y b_z - a_z b_y)\vec{i} + (a_z b_x - a_x b_z)\vec{j} + (a_x b_y - a_y b_x)\vec{k} \ .$$

- **Scalar Triple Product**:
  $\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{b})$.

- **Lagrange's formula** (Vector Triple Product):
  $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$.

- Kinematics is the study of motion, without inquiring about the causes of the motion.
- Let the vector $\vec{x}(t)$ represent the position of a particle at time t, and $\Delta\vec{x}$ its **displacement** in the time interval Δt. (For one-dimensional problems, $\vec{x}$ may be simply written as x).
- Its average velocity during that interval is then $\dfrac{\Delta\vec{x}}{\Delta t}$. The instantaneous **velocity** is obtained in the limit $\Delta t \to 0$, and is denoted in calculus notation by $\vec{v} = \dfrac{d\vec{x}}{dt}$.
- The **acceleration** is $\vec{a} = \dfrac{d\vec{v}}{dt} = \dfrac{d^2\vec{x}}{dt^2}$.
- If the **distance** moved in the time interval Δt is D, then the average **speed** is $\dfrac{D}{\Delta t}$.

Did You Know?

A particle moving at constant (non-zero) speed around a circle has zero average velocity on completing one round.

- A **set** is a collection of items called **elements**. For example, the set $C = \{2, 4, 6\}$ consists of the three elements 2, 4, and 6. The number of elements of a set A is denoted by $n(A)$.
  The **empty set** is denoted by $\emptyset$.
  $x \in A$ means 'x is an element of set A'.

- Two sets are equal if they have the same elements. Set A is a **subset** of set B, denoted by $A \subset B$, if every element of A is also in B.

- The **universal set**, $\mathcal{E}$, consists of all those elements under consideration. The **complement** of A, denoted by A', includes all elements of $\mathcal{E}$ not in A. For example, in the set of positive integers, the complement of the set of odd integers is the set of even numbers (including zero).

- **Union of Sets**: $A \cup B = \{x | x \in A \text{ or } x \in B\}$.

- **Intersection of Sets**:
  $A \cap B = \{x | x \in A \text{ and } x \in B\}$.

- $A \cap A' = \emptyset$.

- $A \cup A' = \mathcal{E}$.

- $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

- $(A \cap B)' = A' \cup B'$ and $(A \cup B)' = A' \cap B'$.

- The **probability** of an event A occurring is denoted by $P(A)$ with $0 \leq P(A) \leq 1$.

- The value of $P(A)$ may be estimated by noting the **relative frequency** of the occurrence of A in repeated trials of an experiment, or in accumulated data of observations.

- Given $p = P(A)$, then n repeated trials of the same experiment will yield event A np times on average. That is, **expectation** $E(A) = np$.

- **Independent Events**: (See the 'Sets' chapter for the $\cap$ and other notation). $P(A \cap B) = P(A) \times P(B)$.

- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

- **Conditional Probability**: The probability of event B given that event A has occurred is $P(B|A) = \dfrac{P(A \cap B)}{P(A)}$.

- **Bayes' Theorem**:

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)},$$

where $P(A)$ may be written as $P(A|B)P(B) + P(A|B')P(B')$ with B' being the complement of B; that is $P(B) + P(B') = 1$.

For a data set x_i, $i = 1, 2, \dots, n$,

- The **mean** μ is defined by

$$\mu = \frac{1}{n}\sum_{i=1}^{n} x_i \ .$$

- The **variance** is

$$\begin{aligned}\sigma^2 &= \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu)^2 \\ &= \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \mu^2 \ ,\end{aligned}$$

  where σ is the **standard deviation**.

- **Bhatia-Davis Inequality:**
  For a bounded probability distribution $P(X)$ with $m \leq X \leq M$, we have

$$\sigma^2 \leq (M - \mu)(\mu - m).$$

- **Samuelson's Inequality**:
  For each x_i,

$$\mu - \sigma\sqrt{n-1} \leq x_i \leq \mu + \sigma\sqrt{n-1}.$$

- A **function** f maps an element x of a **domain** set to a unique element $y = f(x)$ in its **codomain**.
- The **range** or **image** of the function is a subset of the codomian.
- A function f is **one-to-one** (injective) if $f(a) = f(b)$ implies $a = b$.
- A function f is **onto** (surjective) if its range coincides with its codomain.
- A function that is both injective and surjective is called **bijective**.
- The **inverse function** f^{-1} reverses the mapping due to f. $f^{-1}(y) = x$ where $y = f(x)$. (f^{-1} exists if f is one-to-one, or if the domain of f^{-1} is suitably restricted.)
- Two functions f and g can be **composed** to give a new function $g \circ f$ that acts as $g \circ f(x) = g(f(x))$.

Did You Know?

Jacobi's identity for vectors:

$$\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = 0.$$

Classical computers operate according to Boolean logic. Boolean algebra implements Boolean logic using rules described below.

- In Boolean algebra a variable x takes only one of two values to represent true/false states: 1 (TRUE) or 0 (FALSE).

- **AND** operation: Denoted by $\cdot$ or $\wedge$.
$x \wedge y = 1$ if and only if $x = y = 1$; otherwise it equals 0. This operation is commutative, $x \wedge y = y \wedge x$, and associative, $x \wedge (y \wedge z) = (x \wedge y) \wedge z$.

- **OR** operation: Denoted by $+$ or $\vee$.
$x \vee y = 0$ if and only if $x = y = 0$; otherwise it equals 1. This operation is also commutative and associative.

- **NEGATION**: Denoted by $\bar{x}$ or x'.
$\bar{x} = 1 - x$ where " $-$ " here denotes the usual subtraction.

- **Distributive Laws**:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

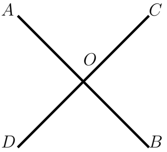
and $$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

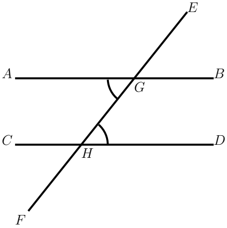
- **Deduction**: Proceeds from the initial statement to the conclusion through a sequence of logical steps.
- **Contradiction**: A method of proving the truth of a statement by first assuming its negation, and showing that that leads to a contradiction. For example, this method is usually used to show that $\sqrt{2}$ is irrational.
- **Contrapositive**: The statement $A \Rightarrow B$ is logically equivalent to the statement $\bar{B} \Rightarrow \bar{A}$, where $\bar{B}$ is the negation of B. For example, "all even numbers are divisible by 2" is equivalent to "if a number is not divisible by 2, it cannot be even".
- **Induction**: The truth of a formula for all natural numbers n is determined by first showing the formula to be true for $n = 1$, and then showing that its presumed truth for $n = k$ implies its truth for $n = k + 1$.

Did You Know?

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

- **Parallel lines** do not meet, even when extended. **Perpendicular lines** meet at an angle of 90°.
- If lines AB and CD intersect at O, then $\angle AOC = \angle BOD$ ('**vertically opposite angles**').
- If P is a point not on the line AB produced, then the **shortest distance** from P to AB produced is along the perpendicular from P to that line.
- Let AB and CD be two parallel lines intersected by the line EF at G and H respectively (with A and C on the same side of EF). Then $\angle AGH = \angle DHG$ ('**alternate angles**').





- The straight line joining points (x, y) and (x_1, y_1) is
$$\frac{y - y_1}{x - x_1} = m = \tan\alpha\,,$$
where $0 \le \alpha < \pi$ is the angle the line makes with the positive x-axis, and m is the **slope** (**gradient**). The equation may be re-written as $y = mx + c$ where c is the **intercept** on the y-axis.

- If another line is **perpendicular** to the above line, its slope must be $-1/m$.

- The **mid-point** of a line joining two points (x_1, y_1) and (x_2, y_2) is $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$.

- Given the vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$, of a triangle, with their relative order being anti-clockwise, the **area of the triangle** is

$$\begin{aligned} A &= \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{vmatrix} \\ &= \frac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1 - x_1y_3 - x_3y_2 - x_2y_1) \end{aligned}$$

The terms are generated as follows: Start at the left-edge of the top row and multiply each term in the top row by a term one step to the right in the bottom row, adding the pieces. Then start at the right-edge of the top row and multiply each term in the top row by a term one step to the left in the bottom row, adding the pieces. Finally, subtract those two contributions and include the overall 1/2.

- For a **polygon** with n sides the general **shoelace algorithm** for the area is

$$A = \frac{1}{2}|x_1y_2 + x_2y_3 + \cdots + x_{n-1}y_n + x_ny_1 - x_2y_1 - x_3y_2 - \cdots - x_ny_{n-1} - x_1y_n| \, .$$

Did You Know?

The book *Real World Mathematics*, by W. K. Ng and R. Parwani, contains questions on real-world applications of most of the topics in this handbook.

Check it out at
www.simplicitysg.net/books/rwm

- A line joining two points on the circumference of a circle forms a **chord**. The perpendicular line from the centre of the circle to the chord bisects the chord (conversely, the perpendicular bisector of a chord passes through the centre).

- **Uniqueness**: Given any three points not on a line, exactly one circle passes through those points.

- A **tangent** to a circle at any point P on its circumference is perpendicular to the line OP where O is the centre of the circle. Therefore, the **normal** to the circle at P lies along OP.

- The **equation for a circle** is

$$(x - x_0)^2 + (y - y_0)^2 = r^2 ,$$

where (x_0, y_0) is the centre and r the radius.

- Given the form

$$x^2 + y^2 - 2ax - 2by + c = 0 ,$$

for some constants a, b, c, one can complete the squares to get $(x-a)^2+(y-b)^2 = a^2+b^2-c$; if $c < a^2+b^2$ then the equation represents a circle with centre (a, b) and radius $R = \sqrt{a^2 + b^2 - c}$.

- **Inscribed Angle Theorem**: Let A and B be two points on the circumference of a circle. The angle subtended by A and B at the centre of the circle is twice that subtended at a point C on the circumference, see Fig.(29.1). This implies that two angles subtended by the same chord, on its same side, must be equal.

- **Tangent-Chord Theorem (Alternate Segment Theorem)**: Let the triangle ABC be inscribed in a circle and a tangent drawn at point A. Let D be another point on the tangent line such that D and C are on opposite sides of the line AB. Then $\angle DAB = \angle BCA$. See Fig.(29.2).

- **Intersecting Chords Theorem**: Let A, B, C, and D be points on the circumference of a circle and X the point of intersection of the lines AC and BD. Then the triangle ABX is similar to triangle DCX. See Fig.(29.3). This implies $AX \cdot CX = BX \cdot DX$.

- **Tangent-Secant Theorem**: Let A, B and C be points on the circumference of a circle and let the tangent at A meet the line CB produced at D. Then $(DA)^2 = DB \times DC$. See Fig.(29.4).

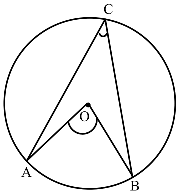


Figure 29.1: Figure for Inscribed Angle Theorem.

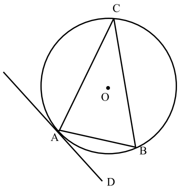


Figure 29.2: Figure for Tangent-Chord Theorem.

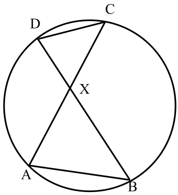


Figure 29.3: Figure for Intersecting Chords Theorem.

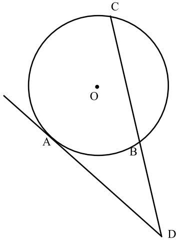
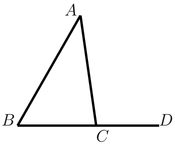
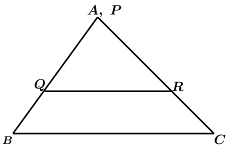


Figure 29.4: Figure for Tangent-Secant Theorem

- In triangle ABC let BC be produced to D as shown below. Then the exterior angle $ACD = \angle BAC + \angle ABC$.
- In triangle ABC let a, b and c represent the sides opposite the corresponding angles (vertices) denoted in upper-case. Then $A > B \Leftrightarrow a > b$.
- In a triangle ABC with non-zero area, $a+b > c$.
- **Pythagoras' Theorem**: In triangle ABC with $C = 90^\circ$, $a^2 + b^2 = c^2$.
- Two triangles ABC and PQR are **similar** if one is a scaled version of the other; see figure below. That is, if they have the same angles, then their sides are in the same proportion: If $QR \| BC$, then $BC/QR = AB/PQ$.
- **Heron's formula** for the area of a triangle: Area $= \sqrt{s(s-a)(s-b)(s-c)}$ where $2s = (a+b+c)$.





Two triangles are congruent (identical), if they satisfy any one of the conditions listed below. S refers to a side while A to an angle.

- SSS. That is, all three sides are the same for both triangles.
- SAS. Two sides and the included angle are the same for both triangles.
- ASA. One side and the two angles adjacent to that side are the same for both triangles.
- Other cases:
  (i) AAS reduces to the ASA case since the sum of angles in a triangle is 180°.
  (ii) In a right-angled triangle, the sides and angles are constrained, so it is easy to check for congruence.

Did You Know?

Given a triangle with perimeter P and area A, we have the inequality $P^2 \geq 12\sqrt{3}A$, with equality holding for equilateral triangles.

- **Centroid theorem**:
  In $\triangle\ ABC$ let P, Q and R be mid-points of the sides AB, BC and CA respectively. Then the medians CP, AQ and BR pass through a point G (**centroid**) inside the triangle, and $GP = CP/3$, with similar relations for the other bisectors.

- **Circumcentre theorem**:
  In $\triangle\ ABC$ let PP', QQ' and RR' be perpendicular bisectors of the sides AB, BC and CA respectively. The bisectors pass through a point O (**circumcentre**), which may lie outside the triangle. A circle drawn with O as centre circumscribes the triangle.

- **In-centre theorem**:
  In $\triangle\ ABC$ let AP, BQ and CR be bisectors of the angles A, B and C respectively. The bisectors pass through a point I (**incentre**) inside the triangle. A circle drawn with I as centre can be inscribed in the triangle.

- The **Euler line**, connecting the centroid and circumcentre, passes through the **orthocentre**, which is the point of intersection of the three altitudes of the triangle. An **altitude** is a line from a vertex that is perpendicular to the opposite side.

- There are five regular convex polyhedra in three dimensional space (**Platonic Solids**). 'Regular' means that the faces of a polyhedron are identical. The number of faces F, edges E, and vertices V is indicated below for each case.
- Tetrahedron: $F = 4, \ E = 6, \ V = 4$.
- Hexahedron: $F = 6, \ E = 12, \ V = 8$.
- Octahedron: $F = 8, \ E = 12, \ V = 6$.
- Dodecahedron: $F = 12, \ E = 30, \ V = 20$.
- Icosahedron: $F = 20, \ E = 30, \ V = 12$.
- Each case satisfies **Euler's polyhedron formula**, $V - E + F = 2$, which holds more generally for non-regular polyhedra. (The boundary of a convex polyhedron can be deformed into the surface of a sphere. The '2' in the formula above is the **Euler characteristic** of a sphere).
- There is a duality between the solids, in the exchange $V \longleftrightarrow F$.
- Let G be a **planar graph**, that is, a collection of vertices V on the plane connected by edges E. If G is **connected**, that is, there is a path between any two vertices, then Euler's formula above applies to the graph with F counting the faces (including the outer region as one face).

- For angle measurements in **radian**, $\theta \equiv s/R$ where s is the arc length of circle, of radius R, subtended by that angle. Therefore 2π radians equals $360°$.

- For a **right-angled triangle** ABC, with $C = 90°$, $\sin A = a/c$, $\cos A = b/c$ and $\tan A = \sin A/\cos A = a/b$, where the small case letters denote lengths opposite the corresponding angles.

- The sin and cos functions **range** over the interval $[-1, 1]$ while the tan function ranges over the real line.

- **Periodicity**: $\sin(\theta + 360°) = \sin(\theta)$, $\cos(\theta + 360°) = \cos(\theta)$ and $\tan(\theta + 180°) = \tan(\theta)$.

- The **principal values** for the **inverse functions** $\sin^{-1}$ and $\tan^{-1}$ are those that lie in the range $-\pi/2 \leq y \leq \pi/2$, while the $\cos^{-1}$ function has the range $0 \leq y \leq \pi$.

- The functions csc, sec and cot are **reciprocals** of the sin, cos and tan functions ($\csc x$ is also written as cosec x).

- Phenomena that are described by an equation of the form $y(x) = A + B\sin(kx + C)$ are called **sinusoidal**.

- **Some identities**:

$$\sin^2 A + \cos^2 A = 1.$$

$$\sin(-A) = -\sin A \quad \text{and} \quad \cos(-A) = \cos A.$$

$$\tan A = \sin A / \cos A \quad \text{and} \quad \tan(-A) = -\tan A.$$

- **Addition Formulae**:

$$\sin(A \pm B) = \sin A \ \cos B \pm \cos A \ \sin B \ .$$

$$\cos(C \pm D) = \cos C \ \cos D \mp \sin C \ \sin D \ .$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \ \tan B} \ .$$

- **Factor Formulae**:

$$\sin A \pm \sin B = 2 \sin \frac{A \pm B}{2} \cos \frac{A \mp B}{2} \ .$$

$$\cos A + \cos B = 2 \cos \frac{A + B}{2} \cos \frac{A - B}{2} \ .$$

$$\cos A - \cos B = -2 \sin \frac{A + B}{2} \sin \frac{A - B}{2} \ .$$

- **R-Formulae**: With $R = \sqrt{a^2 + b^2}$ and $\tan \alpha = b/a$,

$$a \cos \theta \pm b \sin \theta = R \cos(\theta \mp \alpha) \ .$$

$$a \sin \theta \pm b \cos \theta = R \sin(\theta \pm \alpha) \ .$$

- The **sides** of a triangle ABC are labelled with small case letters a, b and c denoting lengths opposite the corresponding **angles** A, B and C.

- The **sine rule**:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = 2R \ ,$$

where R is the radius of the circle that circumscribes the triangle (the centre of the circle lies at the intersection of the perpendicular bisectors of the three sides).

- The **cosine rule**:

$$c^2 = a^2 + b^2 - 2ab \ \cos C \ .$$

- The **area** of the triangle is

$$\text{Area} = \frac{1}{2} ab \sin C \ .$$

Did You Know?

The overdot notation $\dot{x}$ to denote $\dfrac{dx}{dt}$, and $\ddot{x}$ for $\dfrac{d^2x}{dt^2}$, was invented by Newton.

- If a point P is above (below) the horizontal through the observation point O, then the angle that OP makes with the horizontal is the **angle of elevation** (**angle of depression**) of P.

- An **absolute bearing** denotes a direction relative to North. It is usually expressed by a clockwise angle measured in degrees, for example, 065°.

- **Relative bearings** define an angle relative to a chosen axis, for example the axis along which an aircraft is pointing.

Did You Know?

L'Hopital's Rule: If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if the later limit exists. For example,

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = 1.$$

- Let Δf represent the change in a function $f(x)$ as x changes by Δx. The **derivative** of f is defined by $\dfrac{df}{dx} = f'(x) \equiv \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$. (We assume here and below that the function is twice differentiable).

- The **stationary** points of $y = f(x)$ are those for which $\dfrac{df}{dx} = 0$. They are **turning points** (**local minima** or **local maxima**), or **points of inflexion** (where the first derivative has the same sign on both sides of the point).

- If $\dfrac{d^2 f}{dx^2} > \ (<)\ 0$ at the stationary point, it is a local minimum (maximum). Stationary points with $\dfrac{d^2 f}{dx^2} = 0$ can be examined by checking the sign of $f'(x)$ on either side of the point.

- The **global extrema** occur at the boundaries of the domain or at stationary points.

- If Δx is small but not strictly zero, we may form the approximation

$$\Delta y \approx \left(\frac{dy}{dx} \right) \times \Delta x \,.$$

Let f and g be functions of x, and A, B, n represent constants in the following.

- $\dfrac{d}{dx}Ax^n = Anx^{n-1}$.
- $\dfrac{d}{dx}e^x = e^x$.
- $\dfrac{d}{dx}\ln x = \dfrac{1}{x}$.
- $\dfrac{d}{dx}\sin x = \cos x$ and $\dfrac{d}{dx}\cos x = -\sin x$.
- $\dfrac{d}{dx}\tan x = \sec^2 x$.
- $\dfrac{d}{dx}(f+g) = f' + g'$.
- $\dfrac{d}{dx}(fg) = fg' + gf'$.
- $\dfrac{d}{dx}\dfrac{f}{g} = \dfrac{gf' - fg'}{g^2}$.
- $\dfrac{d}{dx}f(g(x)) = \dfrac{df}{dg} \times \dfrac{dg}{dx}$.
- $\dfrac{d}{dx}f = 1/\left(\dfrac{dx}{df}\right)$.

- **Fundamental Theorem of Calculus**:

$$\int_a^b f(x)\,dx = F(b) - F(a)\,,$$

where $F(x)$ is a function that satisfies

$$\frac{dF(x)}{dx} = f(x)\ .$$

- Hence $\int_a^b \frac{dF}{dx}\,dx = F(b) - F(a)$.

- **Indefinite integrals**: $\int f(x)\,dx = F(x) + C$, where F and f are related as above, and C is a constant of integration which can be fixed once we have more information about the problem.

- For calculating **areas** between the curve $y = f(x)$ and the x-axis through the formula $\int y\,dx$, note that if $f(x) < 0$ within a region $x_1 \le x \le x_2$, the integral in that region would give a negative value and the area there is then the negative of the integral.

- One may also evaluate the area between a curve and the y-axis. In this case the integral would be $\int_{y_1}^{y_2} x\,dy$.

In the formulae below, f and g are functions of x while a, b, n are constants; C is a constant of integration.

- $\int (a+bx)^n \; dx = \frac{(a+bx)^{n+1}}{b(n+1)} + C, \quad n \neq -1.$
- $\int \frac{1}{a+bx} \; dx = \frac{1}{b} \ln(a+bx) + C.$
- $\int \sin(a+bx) \; dx = \frac{-1}{b} \cos(a+bx) + C.$
- $\int \cos(a+bx) \; dx = \frac{1}{b} \sin(a+bx) + C.$
- $\int e^{(a+bx)} \; dx = \frac{1}{b} e^{(a+bx)} + C.$
- $\int (f+g) \; dx = \int f \; dx + \int g \; dx.$
- $\int_a^c f(x) \; dx = \int_a^b f(x) \; dx + \int_b^c f(x) \; dx.$

Did You Know?

$$\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$$

Some long standing conjectures were only proven true relatively recently:

- **Kepler's Conjecture** (proven)
  The way to pack equal sized spheres in three-dimensional space, so as to maximise the average density, is the intuitive regular arrangement!
- **The Four Colour Map Theorem**
  No more than four colours are sufficient to colour a map of contiguous countries on the plane, without adjacent countries having the same colour.
- **Fermat's Last Theorem**
  The equation $x^n + y^n = z^n$ has no positive integral solution (x, y, z) for any integer $n > 2$.

Did You Know?

There are an infinite number of integral solutions corresponding to Pythagoras' Theorem, $x^2 + y^2 = z^2$.

Explicitly,
$x = m^2 - n^2,\ y = 2mn, z = m^2 + n^2$,
where m, n are any positive integers.

Many mathematicians believe the conjectures below to be true, but rigorous proofs are lacking at the time of writing.

- **Goldbach Conjecture**
  Every even integer larger than 2 can be written as the sum of two primes. For example, $42 = 5 + 37$.

- **Twin Prime Conjecture**
  There are infinitely many twin primes. (A 'twin' prime differs from another prime number by 2. For example, 11 and 13 are twin primes.)

- **Collatz Conjecture**
  Start with any positive integer n. If it is even, apply the rule $n \to n/2$, while if it is odd apply the rule $n \to 3n + 1$. Continue the iteration on each result. You will eventually reach the number 1. For example, $13 \to 40 \to 20 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1$.

- **Beal's Conjecture**
  Let $x^m + y^n = z^p$, with all letters representing positive integers, and m, n, p each being greater than 2. Then, if x, y, z are pairwise relatively prime, the equation has no solutions. (Note: This conjecture is a generalisation of Fermat's Last Theorem).